

## S-duality in $N = 2$ supersymmetric gauge theories

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**ABSTRACT:** A solution to the infinite coupling problem for  $N = 2$  conformal supersymmetric gauge theories in four dimensions is presented. The infinitely-coupled theories are argued to be interacting superconformal field theories (SCFTs) with weakly gauged flavor groups. Consistency checks of this proposal are found by examining some low-rank examples. As part of these checks, we show how to compute new exact quantities in these SCFTs: the central charges of their flavor current algebras. Also, the isolated rank 1  $E_6$  and  $E_7$  SCFTs are found as limits of Lagrangian field theories.

**KEYWORDS:** Supersymmetry and Duality, Duality in Gauge Field Theories, Supersymmetric gauge theory, Conformal and W Symmetry.

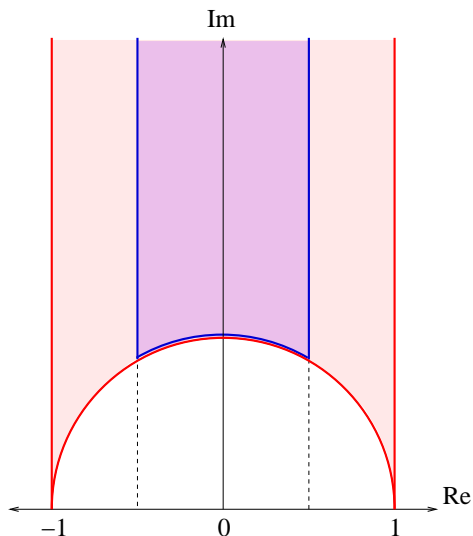
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**1. Infinite coupling and S-duality**

In many  $N = 2$  supersymmetric gauge theories in four dimensions, an exactly marginal gauge coupling,  $g$ , can be taken infinite. In this paper we propose a new kind of quantum equivalence of gauge theories which relates such infinitely-strongly coupled theories to ones with both weakly-coupled ( $g \ll 1$ ) and strongly-coupled ( $g \sim 1$ ) sectors, but no infinitely-strongly coupled sectors. This proposal thus allows one to eliminate infinitely-coupled gauge theories in favor of merely strongly-coupled ones. In particular, it suggests that even as  $g \rightarrow \infty$  all the correlation functions of the theory remain finite! Our proposal generalizes the well-known S-duality of  $N = 4$  supersymmetric gauge theories to the larger class of  $N = 2$  supersymmetric ones.

S-duality, or Olive-Montonen duality [1], in  $N = 4$  supersymmetric gauge theories in four dimensions answers the question of what happens as the gauge coupling constant becomes infinite: the theory actually becomes a weakly coupled gauge theory again, though

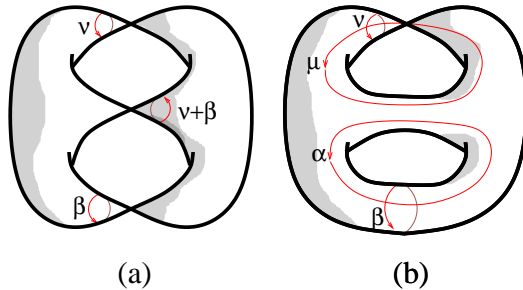


**Figure 1:** Fundamental domain in  $\tau$  for  $SL(2, \mathbb{Z})$  (blue), and in  $\tilde{\tau}$  for  $\Gamma^0(2)$  (red). Edges of the domains are identified under reflection through  $\text{Re } \tau = 0$ .

not necessarily with the same gauge group. In theories with simply-laced gauge group, where the theory is self-dual, this is expressed as the equivalence between the theory at different couplings,  $\tau \simeq -1/\tau$ , where  $\tau = \theta/2\pi + 4\pi i/g^2$  is the complex coupling. Combined with the angularity of the theta angle,  $\tau \simeq \tau + 1$ , this generates an  $SL(2, \mathbb{Z})$  group of identifications whose fundamental domain in the space of couplings is bounded away from infinite coupling ( $\text{Im } \tau = 0$ ); see figure 1.

But this is not always the answer to the infinite coupling problem in scale-invariant gauge theories with less supersymmetry. Though in the case of  $N = 2$   $SU(2)$  superQCD with four massless fundamental hypermultiplets there is an  $SL(2, \mathbb{Z})$  S-duality [2], there are higher-rank gauge theories which are not self-dual. For example, for  $N = 2$   $SU(3)$  with six massless fundamentals, the S-duality group is  $\Gamma^0(2) \subset SL(2, \mathbb{Z})$  generated by [3]  $\tilde{\tau} \simeq \tilde{\tau} + 2$  and  $\tilde{\tau} \simeq -1/\tilde{\tau}$  where  $\tilde{\tau} := 2\tau$ . The fundamental domain of this group in the coupling space is not bounded away from infinite coupling, but instead contains the point  $\tilde{\tau} = 1$ , as shown in figure 1.

This raises the question of how to characterize the physics at the infinite coupling point. The simplest possibility is that this limit is actually a different weakly-coupled  $N = 2$  gauge theory, but the exact low energy effective action shows that this cannot be the case. One way to see this is to compare the behavior of the curve [2] encoding the Coulomb branch effective action in the weak and infinite coupling limits. As  $\text{Im } \tilde{\tau} \rightarrow \infty$  in the  $SU(3)$  theory, three non-intersecting cycles pinch in the genus 2 curve at any point on the moduli space; see figure 2a. (More precisely, we are picking out the particular vanishing cycles for which there exist BPS states in the spectrum.) Each pinching cycle is the signature of a pair of charged  $W^\pm$  gauge bosons becoming massless, corresponding to the expected 6 gauge bosons given a mass by the  $SU(3) \rightarrow U(1) \times U(1)$  Higgsing. In contrast, as  $\tilde{\tau} \rightarrow 1$ , the



**Figure 2:** Degenerations of the scale invariant  $SU(3)$  curve at a generic point on the Coulomb branch at (a) weak coupling, and (b) infinite coupling.

infinite coupling point, only one cycle vanishes at generic points on the moduli space. This does not give enough  $W$  bosons to account for a weakly-coupled Higgs mechanism. So, some new phase is indicated for this  $N = 2$   $SU(3)$  gauge theory at  $\tilde{\tau} = 1$ . This type of behavior of the Coulomb branch effective action is typical of the infinite-coupling points of many infinite series of such theories [4].

To avoid confusion, we should note that the numerical value of  $\text{Im} \tilde{\tau}$  is not really indicative of whether a point in coupling space is such an infinite-coupling point or not. For, lacking any other non-perturbative definition of the coupling, one could always make a holomorphic, non-perturbative redefinition of the coupling to change its value at the putative infinite-coupling point to any desired value. Indeed, in the rest of the paper we will find it convenient to describe couplings in terms of a function  $f(\tilde{\tau})$  which approaches  $f \sim e^{i\pi\tilde{\tau}} \rightarrow 0$  at weak coupling, but  $f \rightarrow 1$  at “infinite coupling”.

We nevertheless use “infinite coupling” as a convenient phrase to describe those points in coupling space where the effective action has the singular (but not weak-coupling) behavior at generic points on the Coulomb branch described above. These infinite coupling points can be invariantly characterized as follows. Abstractly, the space of couplings is a complex manifold with singularities, and the S-duality group is the fundamental group in the orbifold sense [11] of this space with the singular points removed. The infinite coupling points we are interested in are the cusps, i.e. the points where the S-duality identification is of infinite order (like the  $\tau \rightarrow \tau + 1$  theta angle identification at weak coupling). The coupling is just a complex coordinate on this space. If the coupling  $\tau$  transforms by fractional linear transformations under the S-duality group, then  $g^2 \sim (\text{Im} \tau)^{-1}$  is either 0 or infinite at cusps.

In this paper we argue that the physics at the infinite coupling limit of a scale-invariant  $N = 2$  gauge theory with gauge group  $H$ ,  $\text{rank}(H) = r$ , is a *weakly coupled* scale-invariant gauge theory with gauge group  $G$  with smaller rank,  $\text{rank}(G) = s < r$ , which is coupled to an isolated rank  $(r - s)$   $N = 2$  superconformal field theory.<sup>1</sup> Here “isolated” means that

<sup>1</sup>The rank of an isolated  $N = 2$  SCFT, where an explicit gauge group and Lagrangian description are lacking, is defined as the complex dimension of its Coulomb branch, which is the number of  $U(1)$  gauge factors generically unbroken in the IR.

this SCFT has no exactly marginal coupling of its own. Thus, the SCFT can be thought of as providing “matter fields” charged under  $G$ . More precisely, in the infinite coupling limit,  $G$  weakly gauges a subgroup of the flavor symmetry of the SCFT. (We will always use “flavor symmetry” to refer to the global symmetry which commutes with the  $N = 2$  superconformal symmetry.)

In other words, we are proposing a generalization of S-duality from  $N = 4$  to  $N = 2$  scale-invariant field theories. Our proposal of including strongly-coupled  $N = 2$  SCFTs as factors in the duals of non-Abelian gauge theories is a natural generalization of the  $N = 4$  case, given the existence of isolated  $N = 2$  conformal gauge theories.

Let us describe the infinite-coupling duality more precisely. Denote a gauge theory with gauge group  $G$  and matter half-hypermultiplets in the  $\bigoplus_i \mathbf{r}_i$  representation by

$$G \text{ w/ } \bigoplus_i \mathbf{r}_i, \tag{1.1}$$

where w/ is read as “with”. Now consider a theory described by a SCFT with flavor symmetry group  $S$ , a subgroup  $G \subset S$  of which is gauged. We will denote such a theory in a similar manner as

$$G \text{ w/ SCFT}_S, \tag{1.2}$$

emphasizing that the SCFT acts as “matter” for the gauge theory (even though  $N = 2$  SCFTs also have gauge degrees of freedom). Note that  $S$  is *not* the flavor symmetry of this theory, except in the limit of zero coupling; at finite coupling only the subgroup of  $S$  commuting with  $G$  is a global symmetry. In general there may be more than one way of embedding  $G$  in  $S$ . A given embedding can often be specified by the maximal subgroup  $G \times F \subset S$ . Gauging  $G$  leaves  $F$  as the flavor group.

In the rest of this paper we will give evidence for specific examples of the duality between infinitely-coupled Lagrangian  $N = 2$  SCFTs and gaugings of isolated SCFTs, as described above. For example, in the next section we argue that

$$\text{SU}(3) \text{ w/ } 6 \cdot (\mathbf{3} \oplus \bar{\mathbf{3}}) = \text{SU}(2) \text{ w/ } (2 \cdot \mathbf{2} \oplus \text{SCFT}_{E_6}). \tag{1.3}$$

In words and more detail: the scale-invariant theory with gauge group  $G = \text{SU}(3)$  coupled to 6 massless fundamental hypermultiplets with coupling  $f$  (reviewed in appendix A) is equivalent to an  $\text{SU}(2)$  gauge theory with one massless fundamental hypermultiplet and coupled to the isolated rank 1 SCFT with flavor symmetry  $E_6$  (reviewed in appendix B) by gauging the  $\text{SU}(2)$  in the maximal subgroup  $\text{SU}(2) \times \text{SU}(6) \subset E_6$  with coupling  $\tilde{f}$ . The map between the direct and dual couplings,  $\tilde{f}(f)$ , is given in (2.7), and maps infinite coupling in  $f$  to zero coupling in  $\tilde{f}$ .

Two immediate checks of this proposal are that the ranks and flavor groups of the two sides of (1.3) match. The rank (or dimension of the Coulomb branch) of the  $\text{SU}(3)$  theory is 2, while the  $\text{SU}(2)$  and  $E_6$  SCFT factors on the right are each rank 1. The flavor groups match since that on the left is  $\text{U}(6)$ , while the  $2 \cdot \mathbf{2}$  factor on the right contributes a  $\text{U}(1)$  and the  $E_6$  SCFT contributes a  $\text{SU}(6)$  because of the way the gauged  $\text{SU}(2)$  factor is embedded in  $E_6$ . This embedding must also be consistent with the low energy effective action, giving independent evidence for this proposal, described in section 2.

It is less trivial to see that the number of marginal couplings is the same on each side of (1.3). Clearly there is only one marginal coupling on the left side, but the  $SU(2)$  coupling on the right will be marginal only if the contribution of the rank 1 SCFT “matter” to the  $SU(2)$  gauge coupling beta function has the correct value. This contribution is governed by the central charge of the flavor current algebra of the rank 1 SCFT. By weakly gauging the global flavor symmetries on both sides of the duality and comparing its gauge coupling beta functions we can independently compute this central charge, and verify that the  $SU(2)$  gauge coupling is indeed marginal. This is described in detail in section 3, where the same argument is used to give evidence that the  $SU(2)$   $W^\pm$ -bosons are magnetically charged under the  $SU(3)$  gauge group, indicating that the  $SU(2)$  gauge group on the right side of (1.3) is *not* a subgroup of the  $SU(3)$  gauge group on the left side.

An important result of this paper is the above computation of the flavor current algebra central charge of the SCFT. This is a new exactly computed observable of isolated, strongly-coupled  $N = 2$  SCFTs.

A somewhat simpler example is outlined in section 4, where it is shown that

$$\text{Sp}(2) \text{ w/ } 12 \cdot 4 = \text{SU}(2) \text{ w/ SCFT}_{E_7}, \tag{1.4}$$

with the  $SU(2)$  gauging the  $SU(2)$  factor in the maximal embedding  $E_7 \supset SU(2) \times SO(12)$ , to realize the  $SO(12)$  flavor symmetry.

It is worth noting that the conjectures (1.3) and (1.4) identify the  $E_6$  and  $E_7$  rank 1 SCFTs as subsectors of Lagrangian field theories which decouple from the rest of the theory in the infinite coupling limit. The rank 1  $N = 2$  SCFTs with exceptional global symmetry groups [5] have not been previously constructed in a purely four-dimensional field theory framework; instead they have been shown to exist only by dimensional reduction from a 5 or 6 dimensional SCFT, which in turn were constructed as low energy limits of certain string configurations [6].

Other scale invariant rank 2 theories (listed in appendix A.1) have infinite coupling limits whose dual descriptions can be analyzed similarly. We leave this for later, though, since these examples require substantially more work because less complete information is available either about the Coulomb branch effective actions of these theories or about their proposed dual SCFTs. In particular, verification of the same consistency checks as performed for the two examples (1.3) and (1.4) in this paper requires the computation of non-maximal mass deformations of the  $E_n$  rank 1 SCFTs and of the mass-deformed curve of the  $G_2$  w/  $8 \cdot 7$  rank 2 Lagrangian SCFT. The non-maximal mass deformations are mass deformations of SCFTs which have the same conformal curves as the  $E_n$  SCFTs, but have smaller global symmetries than the maximal deformations with  $E_n$  symmetry. Some other examples of identical conformal curves which have inequivalent mass deformations are pointed out in appendix A.

More generally, infinite coupling points are ubiquitous in higher-rank  $N = 2$  Lagrangian theories. For example, all the  $n > 2$   $SU(n)$  theories with  $2n$  fundamental hypermultiplets have infinite coupling limits, and similarly for many other series of scale-invariant theories with rank greater than 2. In fact, the only series of higher-rank theories which are known to have Lagrangian weak coupling descriptions at all the singularities in the space of

marginal couplings, besides the  $N = 4$  theories, are the  $\text{Sp}(n)$  theories with 4 fundamentals and 1 antisymmetric [7, 8], of which  $\text{SU}(2)$  with four fundamentals [2] is a special case, and the  $\text{SU}(2) \times \text{SU}(2)$  theory with 4 fundamentals and 1 bifundamental [9, 10]. However, it is probable that all the  $\text{SU}(2)^n$  cylindrical and elliptic models [11] are also examples of this type.

Finally, we have included several appendices in an attempt to make the paper more self-contained. They mostly either collect scattered results from the  $N = 2$  field theory literature, or review results which are probably known to experts but have not been published.

## 2. Infinite coupling in $\text{SU}(3)_{w/6} \cdot (\mathbf{3} \oplus \bar{\mathbf{3}})$

Our aim is to check the proposed equivalence

$$\text{SU}(3)_{w/6} \cdot (\mathbf{3} \oplus \bar{\mathbf{3}}) = \text{SU}(2)_{w/(2 \cdot \mathbf{2} \oplus \text{SCFT}_{E_6})}. \tag{2.1}$$

In this section we extract evidence supporting it from the known low energy effective action on the Coulomb branch for the  $\text{SU}(3)$  theory.

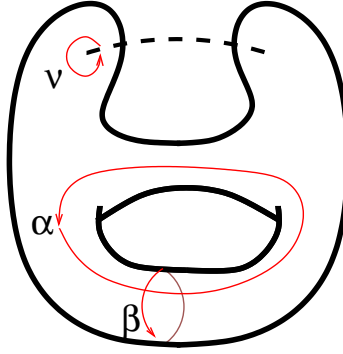
### 2.1 Limits in the effective theory

As reviewed in appendix A.2, the curve encoding the low energy effective action on the Coulomb branch of the  $\text{SU}(3)$  theory with six massless fundamental hypermultiplets is (A.4).  $u$  and  $v$  are the Coulomb branch vevs of dimension 2 and 3, respectively, and their associated basis of holomorphic one-forms are given in (A.3). These one-forms determine the central charges and thus the masses of BPS states. The infinite coupling point is at coupling  $f = 1$ . The curve can be conveniently factorized as

$$y^2 = \left[ (1 - \sqrt{f})x^3 - ux - v \right] \left[ (1 + \sqrt{f})x^3 - ux - v \right]. \tag{2.2}$$

As  $f \rightarrow 1$ , this factorization makes it clear that the curve degenerates to a genus one curve, and it is easy to check that the  $\omega_u = xdx/y$  one-form develops a pair of poles at  $x = \infty$ , whereas  $\omega_v = dx/y$  remains holomorphic. Figure 2b is a representation of this degenerate curve.

This degeneration corresponds to what one expects the curve to be in the weak coupling limit of the right side of (2.1), as we now explain. Recall that mass parameters in  $N = 2$  theories transform in the adjoint representation of the associated global flavor symmetry which they break. A rank one theory with a Coulomb branch vev  $u$  and a single mass parameter  $m$  has an associated rank 1 global flavor symmetry (“quark number”). Its central charge has the form [2]  $Z = ea(u) + ga_D(u) + nm$ , where  $e$  and  $g$  are the electric and magnetic charges of the low energy  $\text{U}(1)$  gauge group, and  $n$  is the quark number. Now imagine weakly gauging the quark number symmetry so that  $n$  becomes the new electric charge of this gauge group and the mass  $m$  becomes the vev of the associated vector multiplet. Thus the central charge obeys  $\partial Z / \partial u = \oint_\gamma \omega_u$  and  $\partial Z / \partial m = \oint_\gamma \omega_m$  for holomorphic one forms  $\omega_{u,m}$  and cycles  $\gamma$  on the associated genus two curve. The weaker this gauging, the



**Figure 3:** Curve of a rank one theory with massive quark. The dotted line marks the pinched handle that unpinches upon weakly gauging the quark number symmetry.

heavier the magnetic monopoles of the quark number  $U(1)$  become, corresponding to a cycle intersecting a vanishing cycle of the degenerating genus two curve, as in the  $\mu$  cycle of figure 2b. When the quark number gauging is turned off, the curve becomes a nondegenerate genus one curve with holomorphic one-form  $\omega_u$ , as shown in figure 3. From the form of the central charge, it follows in this limit that  $\partial Z/\partial m = n = \oint_{\gamma} \omega_m$  for  $[\gamma] = e[\alpha] + g[\beta] + n[\nu]$ , and therefore that the second one-form,  $\omega_m$ , develops a pair of poles.

Thus, the degeneration of the  $SU(3)$  curve when  $f = 1$  may be interpreted as follows:  $v$  becomes the Coulomb branch vev of a rank 1 SCFT in which  $u$  appears as a mass deformation. For  $f$  close to but not equal to 1,  $u$  is the vev of a vector multiplet weakly gauging some global symmetry of this rank 1 SCFT.

Furthermore, the curve of the rank 1 SCFT can be explicitly identified by setting  $u = 0$  and  $f = 1$ . Then the curve (2.2) and holomorphic one-form become

$$y^2 = -v(2x^3 - v), \quad \omega_v = dx/y. \quad (2.3)$$

Making the change of variables  $x = -i\tilde{x}/\tilde{v}$ ,  $y = 2\tilde{y}/\tilde{v}$ ,  $v = 2i\tilde{v}$ , this becomes

$$\tilde{y}^2 = \tilde{x}^3 - \tilde{v}^4, \quad \tilde{\omega}_{\tilde{v}} = d\tilde{x}/\tilde{y}, \quad (2.4)$$

which is the curve of the rank 1  $E_6$  SCFT, reviewed in appendix B.

Also, the curve of the  $SU(2)$  factor which weakly gauges part of the flavor symmetry of the  $E_6$  SCFT can be extracted by setting  $v = 0$  (the conformal point of the  $E_6$  theory) in (2.2),

$$y^2 = x^2 [(x^2 - u)^2 - fx^4], \quad \omega_u = xdx/y. \quad (2.5)$$

The factor of  $x^2$  is due to a pinched cycle at  $x = 0$ . Scaling to the remaining genus one curve by defining  $\hat{y} = y/x$  gives the curve

$$\hat{y}^2 = (x^2 - u)^2 - fx^4, \quad \omega_u = dx/\hat{y}, \quad (2.6)$$

which is precisely the curve [3] of the scale invariant  $SU(2)$   $N = 2$  superQCD. It is weakly coupled at  $f = 0$  and  $f = 1$ , these limits being related by the  $SL(2, \mathbb{Z})$  S-duality of this



theory [2]. In particular, this S duality maps the the curve (2.6) to itself with  $f \rightarrow \tilde{f}(f)$  where  $\tilde{f}(1) = 0$ . Indeed, this strong to weak coupling map can be extracted from [3]:

$$\tilde{f} = 8 \frac{(1-f)(4-9f) + (4-3f)\sqrt{1-f}}{(8-9f)^2}. \tag{2.7}$$

This is an exact relation for the couplings implicitly defined by how they appear in (2.6) and its dual. In the weak coupling limit, this can be related to the traditional gauge coupling defined by  $q = e^{2\pi i\tau} = e^{i\theta} e^{-8\pi^2/g^2}$ . For the SU(2) and SU(3) Lagrangian SCFTs with curves (2.2) and (2.6),  $f \approx -64q$  when  $f$  is near 0 [3]. Thus (2.7) implies that near the infinite coupling point, the dual coupling of the SU(2) factor goes as  $\tilde{q} \approx -(1/2)\sqrt{1-f}$ .

Note that the elements of this duality — that the infinite coupling dual involves the SU(2) and  $E_6$  SCFTs — could have been guessed just from the anomalous dimensions of the chiral operators on the Coulomb branch:  $u$  and  $v$  have dimensions 2 and 3, while the  $E_6$  SCFT is the only rank 1 SCFT with vev of dimension 3, and the SU(2) SCFT is likewise the unique rank 1 SCFT with vev of dimension 2 [5].

## 2.2 Global symmetries

The SU(3) theory with six massless fundamental hypermultiplets has global symmetry group  $SU(6) \times U(1) \times U(2)_R$ . The  $U(2)_R$  R-symmetry is part of the  $N = 2$  superconformal algebra, so is also automatically present in the proposed SU(2) product  $E_6$  SCFT at the infinite coupling point. We now examine how the  $SU(6) \times U(1)$  flavor symmetries are realized in the dual theory.

**The su(6) symmetry.** To be consistent with our proposal that SU(3) with 6 fundamentals near infinite coupling is the  $E_6$  rank 1 SCFT with the SU(2) factor of the maximal  $SU(2) \times SU(6) \subset E_6$  subgroup weakly gauged, it follows that as the  $u$  vev of the SU(2) factor is turned on, this maximal subgroup of  $E_6$  must break as

$$SU(2) \times SU(6) \rightarrow U(1) \times SU(6). \tag{2.8}$$

In terms of the eigenvalues  $m_i$  ( $i = 1, \dots, 6$ ) and  $m$  of the Cartan subalgebra of the  $SU(6) \times SU(2)$  maximal subgroup of  $E_6$  introduced in appendix B.1, the breaking (2.8) corresponds to

$$m \propto \sqrt{u}, \quad m_{1,\dots,6} = 0. \tag{2.9}$$

This can be checked from the low energy effective action as follows. At  $f = 1$ , turn on  $u$  to get the curve

$$y^2 = -(ux + v)(2x^3 - ux - v), \quad \omega_v = dx/y. \tag{2.10}$$

Change variables according to the combined  $SL(2, \mathbb{C})$  transformation and rescalings

$$x = \frac{-2i\tilde{v}(u^2 + 12\tilde{x})}{u^3 + 24\tilde{v}^2 + 12u\tilde{x}}, \quad y = \frac{-1152\tilde{v}^3\tilde{y}}{(u^3 + 24\tilde{v}^2 + 12u\tilde{x})^2}, \quad v = 2i\tilde{v}, \tag{2.11}$$

to bring this into the form

$$\tilde{y}^2 = \tilde{x}^3 - \left( \tilde{v}^2 u + \frac{1}{48} u^4 \right) \tilde{x} - \left( \tilde{v}^4 + \frac{1}{12} \tilde{v}^2 u^3 + \frac{1}{864} u^6 \right), \quad \tilde{\omega}_{\tilde{v}} = \frac{d\tilde{x}}{\tilde{y}}. \quad (2.12)$$

This is a deformed version of the  $E_6$  SCFT curve (2.4). The general mass deformation of the  $E_6$  SCFT is given by (B.2). Comparing to (2.12) gives the  $E_6$  adjoint casimirs

$$M_2 = u, \quad M_5 = 0, \quad M_6 = \frac{u^3}{12}, \quad M_8 = \frac{u^4}{48}, \quad M_9 = 0, \quad M_{12} = \frac{u^6}{864}. \quad (2.13)$$

Note that these coefficients are ambiguous only up to the choice of normalization of  $u$ . Minahan and Nemeschansky [5] determined the  $M_n$  in terms of an explicit basis of  $E_6$  Casimirs. In appendix B.1 we have rewritten these  $E_6$  Casimirs in terms of the Casimirs of its  $SU(2) \times SU(6)$  maximal subgroup. Compare (2.13) to (B.3), (B.4), and (B.5) to identify how turning on  $u$  breaks the  $E_6$  global symmetry. One easily checks that the assignments  $m = \sqrt{u}/2$  and  $m_i = 0$ , consistent with (2.9), reproduce (2.13).<sup>2</sup>

A further test of this embedding of the  $SU(6)$  flavor symmetry in  $E_6$  can be realized by not only turning on the  $u$  vev at  $f = 1$ , but also the fundamental masses  $m_i$  in the  $SU(3)$  theory. From (A.5) the curve at  $f = 1$  is

$$y^2 = -(2u + S_2)x^4 - (2v - S_3)x^3 + (u^2 - S_4)x^2 + (2uv + S_5)x + (v^2 - S_6) \quad (2.14)$$

where the  $S_n$  are the  $SU(6)$  Casimirs introduced in the appendix. The infinite coupling equivalence implies there should exist a change of variables as in (2.11) to bring this to the form of the mass deformed  $E_6$  SCFT curve (B.2). It is too difficult to find this change of variables explicitly, but we can find evidence that it exists by taking the discriminants of the right sides of (2.14) and (B.2) with respect to  $x$ , and comparing. One finds that the two discriminants indeed agree with the identifications

$$v = 2i\tilde{v} + 2i\tilde{S}_3, \quad u = 2\tilde{T} + 4\tilde{S}_2, \quad S_n = (-2i)^n \tilde{S}_n, \quad (2.15)$$

which fix, in addition to the necessary shifts and rescalings of the  $SU(3)$  vevs  $(u, v)$  relative to the masses and vev of the  $E_6$  curve, a rescaling of the  $SU(6)$  mass eigenvalues between

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<sup>2</sup>There are a few subtleties in performing this matching. Some algebra shows that (2.13) is actually consistent with three different mass assignments (up to permutations),

$$\begin{aligned} \text{(a):} \quad & m = \sqrt{u/2}, \quad m_{1,\dots,6} = 0, \quad \iff \mathbf{su}(2) \times \mathbf{SU}(6) \rightarrow \mathbf{u}(1) \times \mathbf{SU}(6), \\ \text{(b):} \quad & m = m_{1,\dots,4} = 0, \quad m_5 = -m_6 = \sqrt{u/2}, \quad \iff \mathbf{su}(2) \times \mathbf{SU}(6) \rightarrow \mathbf{su}(2) \times \mathbf{SU}(4) \times \mathbf{U}(1)^2, \\ \text{(c):} \quad & m = m_{1,2,3} = \sqrt{u/8}, \quad m_{4,5,6} = -\sqrt{u/8}, \quad \iff \mathbf{su}(2) \times \mathbf{SU}(6) \rightarrow \mathbf{u}(1) \times \mathbf{SU}(3)^2 \times \mathbf{U}(1), \end{aligned}$$

where the corresponding adjoint breaking patterns of the  $SU(2) \times SU(6)$  maximal subgroup of  $E_6$  are shown on the right. The (a) breaking, which manifestly leaves an unbroken  $SU(6)$  factor, is the one described above. The (b) breaking pattern actually gives the same picture, since it also leaves an  $U(1) \times SU(6) \subset E_6$  unbroken. This is not manifest because the unbroken  $SU(6)$  does not coincide with the  $SU(6)$  used for the basis of Casimirs. (There are three inequivalent ways of embedding  $SU(2) \times SU(6)$  in  $E_6$ , related by triality of the affine  $\hat{E}_6$  root system. Two of these are related by complex conjugation in  $E_6$ , and so give the same adjoint breaking, the (b) pattern; the third is the (a) pattern.) In particular, in the (b) breaking, the unbroken  $SU(6)$  factor is realized as  $SU(2) \times SU(4) \times U(1)$ . The (c) breaking pattern is different, and corresponds to adjoint breaking of one  $SU(3)$  factor in the  $SU(3)^3 \subset E_6$  maximal subgroup. This does not give the expected global symmetry group of the original  $SU(3)$  SCFT, but is ruled out by the next check.

the two curves by a factor of  $-2i$ . The  $2i$  rescaling of the masses agrees with the one-form rescaling  $\tilde{\omega}_{\tilde{v}} = (\partial v / \partial \tilde{v}) \omega_v = 2i \omega_v$ , since the one-form normalization determines that of the masses through (A.2), and since the  $SU(6)$  is an index one subgroup of  $E_6$ .

**The  $u(1)$  symmetry.** The  $U(1)$  factor of the flavor group of the  $SU(3)$  theory on the left side of (2.1) is realized on the right side as the  $SO(2)$  flavor symmetry rotating the two pseudoreal half-hypermultiplets in the  $\mathbf{2}$  of  $SU(2)$ . So this part of flavor symmetry is realized in terms of weakly coupled degrees of freedom at the infinite coupling point. This is also indicated in the effective action, since the curve (A.5) for the  $SU(3)$  theory with 6 fundamental hypermultiplets (with the  $SU(6)$  masses set to zero) is

$$y^2 = \left[ (x + \sqrt{1-f} M)^3 - u(x + \sqrt{1-f} M) - v \right]^2 - f x^6, \quad (2.16)$$

from which it is apparent that the  $U(1)$  mass deformation vanishes at  $f = 1$ . This behavior is like that at weak coupling where all the mass deformations vanish, and is in accord with the  $U(1)$  flavor symmetry being associated to matter charged only under the  $SU(2)$  gauge group which is weakly coupled at  $f = 1$ .

### 3. Beta functions and central charges

The proposed duality requires the  $SU(2)$  gauge factor on the right side of (2.1) to be scale invariant. This means that the  $2 \cdot \mathbf{2} \oplus \text{SCFT}_{E_6}$  “matter” must contribute just enough to the beta function for the  $SU(2)$  gauge coupling  $\tilde{f}$  to cancel the contribution from the adjoint  $SU(2)$  vector multiplet. The contribution from the  $2 \cdot \mathbf{2}$  half-hypermultiplets follows from the standard perturbative computation (one-loop exact by a non-renormalization theorem), but the contribution from the rank 1  $E_6$  SCFT does not, since the  $E_6$  theory is a strongly coupled theory. Thus demanding the vanishing of the beta function allows us to compute the  $E_6$  SCFT contribution.

We do this in section 3.1, where we also relate this beta function contribution to the central charge,  $k$ , of the  $E_6$  current algebra. They are related because the beta function is proportional to the 2-point function of the gauge currents, and the gauge currents are linear combinations of the  $E_6$  currents since the  $SU(2)$  gauge group is a subgroup of the  $E_6$  flavor group of the SCFT. The central charge  $k$  is analogous to the current algebra central charge familiar from 2-d CFTs, and is a new exactly computable observable of these 4-d SCFTs.

In section 3.2 we give a different way of using the proposed duality to compute  $k$ . In a spirit similar to that of ’t Hooft’s anomaly matching argument [12], one can weakly gauge the flavor symmetry and compute the contribution to its beta function on both sides of the duality. Since at infinite coupling the  $SU(6)$  flavor symmetry is also a subgroup of the  $E_6$  SCFT flavor group, the contribution to its beta function also depends on  $k$ . The agreement of this calculation with the previous one is a non-trivial check of the duality.

We also apply this weak-gauging argument to the  $U(1)$  flavor symmetry to find evidence that the gauge bosons of the  $SU(2)$  dual gauge group are magnetically charged with respect to the original  $SU(3)$  gauge group. Finally, we comment on the application of this argument to the  $R$ -symmetries as well.

### 3.1 Gauge coupling beta function at infinite coupling

The effective gauge coupling is the coefficient of a term quadratic in the gauge fields, so can be computed in a background field formalism by a two point function for the background gauge bosons. The contribution to this correlator from the matter charged under the gauge group is then proportional to the two point function of the conserved current to which the gauge bosons couple. Lorentz and scale invariance and current conservation imply the OPE of currents  $J_\mu^a$  for a (simple) symmetry group  $G$  must have the form

$$J_\mu^a(x)J_\nu^b(0) = \frac{3k}{4\pi^4}\delta^{ab}\frac{x^2g_{\mu\nu} - 2x_\mu x_\nu}{x^8} + \frac{2}{\pi^2}f^{abc}\frac{x_\mu x_\nu x \cdot J^c(0)}{x^6} + \dots \quad (3.1)$$

Here  $k$ , the central charge, is defined relative to the normalization of the structure constants  $f^{abc}$  which are in turn fixed in this paper by choosing the long roots of the Lie algebra to have length  $\sqrt{2}$ . The coefficient of the  $f^{abc}$  term has been chosen so that  $[Q^a, Q^b] = if^{abc}Q^c$ , where  $Q^a := \int d^3x J_0^a$  is the conserved flavor charge, as can be checked by appropriately integrating (3.1). Also the factor of  $3/4\pi^4$  in the central charge term has been chosen to agree with the central charge normalization convention used in [13].

As discussed in appendix D, the central charge for the  $U(n)$  flavor current algebra of  $n$  free half-hypermultiplets is  $k = 1$ , and the central charge for the half-hypermultiplet currents of a weakly gauged subgroup  $G \subset U(n)$  such that  $\mathbf{n} = \oplus_i \mathbf{r}_i$  under  $G$  is

$$k_{G\text{-hypers}} = \sum_i T(\mathbf{r}_i), \quad (3.2)$$

where  $T(\mathbf{r}_i)$  is the quadratic index normalized as in appendix C. This is just the contribution to the beta function of the gauge coupling of the half-hypermultiplets. Thus, the contribution to the central charge of the gauge current algebra by vector multiplets in a gauge group  $G$  will be

$$k_{G\text{-vector}} = -2T(\mathbf{ad}) \quad (3.3)$$

in this normalization, in order to produce the known beta function,  $-2T(\mathbf{ad}) + \sum_i T(\mathbf{r}_i)$ , of the Lagrangian field theory  $G$  w/  $\oplus_i \mathbf{r}_i$ .

Say an isolated SCFT has flavor symmetry  $H$ , and denote the central charge of its flavor current algebra by  $k_H$ . Upon weakly gauging a subgroup of this flavor symmetry,  $G \subset H$ , the arguments of appendix D show that the contribution of the  $H$  “matter” to the gauge current central charge is

$$k_{G \subset H} = I_{G \hookrightarrow H} k_H, \quad (3.4)$$

where  $I_{G \hookrightarrow F}$  is the index of embedding of  $G$  in  $H$ , defined in appendix C.

Putting (3.2)–(3.4) together, the central charge for the  $G$  gauge current algebra — proportional to the coefficient of the beta function for the gauge coupling — in the theory  $G$  w/  $(\oplus_i \mathbf{r}_i) \oplus \text{SCFT}_H$  is  $k_{G\text{-vector}} + k_{G\text{-hypers}} + k_{G \subset H} = -2T(\mathbf{ad}) + \sum_i T(\mathbf{r}_i) + I_{G \hookrightarrow H} k_H$ . For a scale invariant theory the beta function must vanish, giving the central charge of the isolated SCFT flavor algebra as

$$k_H = \frac{2T(\mathbf{ad}) - \sum_i T(\mathbf{r}_i)}{I_{G \hookrightarrow H}}. \quad (3.5)$$

Now apply this to our present example:  $SU(2) \text{ w/ } (2 \cdot \mathbf{2} \oplus \text{SCFT}_{E_6})$ . For  $SU(2)$  in the normalization of appendix C,  $T(\mathbf{3}) = 4$  and  $T(\mathbf{2}) = 1$ . Also, for  $SU(2)$  embedded in  $E_6$  as the  $SU(2)$  factor of the  $SU(2) \times SU(6)$  maximal subalgebra,  $I_{SU(2) \hookrightarrow E_6} = 1$ . Thus we get

$$k_{E_6} = \frac{2T(\mathbf{3}) - 2T(\mathbf{2})}{I_{SU(2) \hookrightarrow E_6}} = 6. \quad (3.6)$$

This is a new, exactly computed observable in the strongly-coupled rank 1  $E_6$  SCFT.

### 3.2 Global symmetry central charges

We can perform a check on this result by comparing the central charges of the flavor current algebras at weak and infinite coupling. They should be the same, since the value of the central charge cannot depend on the coupling. One way to see that is as follows. Above we viewed the mass parameters as vevs of the scalar components of background vector multiplets which gauge the flavor symmetry of the theory. We can further explore these gauge superfields and consider their one loop beta function. The beta function for this flavor coupling is, by a non-renormalization theorem, given exactly by its one-loop contribution, which is independent of the value of the (original) gauge coupling. Thus the flavor beta function, and therefore the central charge, should be the same at both weak and strong coupling. This is similar in flavor to 't Hooft's anomaly matching argument [12], though the comparison here is being made between different values of a marginal coupling instead of between UV and IR scales.

The coupling independence of the central charge can also be seen directly from the structure of representations of the  $N = 2$  superconformal algebra. (Superconformal algebras and their unitary representations are reviewed, for example, in [14].) Conserved flavor currents fall into  $N = 2$  superconformal multiplets whose primary has scaling dimension  $D = 2$ ,  $SU(2)_R$  spin  $I = 1$ ,  $SU(2) \times \widetilde{SU}(2)$  Lorentz spins  $j = \tilde{j} = 0$ , and  $U(1)_R$  charge  $R = 0$ . We denote this primary by  $J_{ij}^a$  where  $a$  labels the generators of the flavor symmetry, while the symmetrized  $ij$  are the  $SU(2)_R$  indices. Thus the flavor current central charge appears in the  $\langle J_{ij}^a(y) J_{kl}^b(z) \rangle$  correlator.

On the other hand, the marginal gauge coupling  $f$  multiplies a chiral superfield term in the Lagrangian,  $f \int d^4\theta \Phi$ . We see this at weak coupling, where  $\Phi = \text{tr}(W^2)$  and  $W$  is the scalar chiral  $N = 2$  vector multiplet field strength superfield. Since terms in the Lagrangian are superconformally invariant,  $\Phi$  must have  $D = 2$ ,  $R = 4$ , and  $I = j = \tilde{j} = 0$ . Thus, the  $f$ -derivative of the flavor current algebra central charge is measured by the three-point function:

$$\frac{\partial k}{\partial f} \sim \int d^4\theta \langle \Phi(x, \theta) J_{ij}^a(y) J_{kl}^b(z) \rangle.$$

But in a superconformal theory, the vacuum expectation of a product of superconformal primaries can only be non-zero if the total  $R$ -charge of the primaries vanishes. Since  $R = 4$  for  $\Phi$ , but vanishes for  $J_{ij}^a$ , we conclude that  $\partial k / \partial f = 0$ , and therefore that the flavor current central charge is independent of the marginal gauge coupling.

**The SU(6) symmetry.** We start with the SU(6) flavor symmetry of the SU(3) w/  $6 \cdot (\mathbf{3} \oplus \bar{\mathbf{3}})$  theory. At weak coupling, the vector multiplets are neutral under the flavor symmetry, and the half-hypermultiplets transform as a  $(\mathbf{3}, \bar{\mathbf{6}}) \oplus (\bar{\mathbf{3}}, \mathbf{6})$  under the  $SU(3) \times SU(6)$  combined gauge and flavor symmetries. Thus the central charge of the flavor current is by (D.6)

$$k_{\text{SU}(6)\text{-weak}} = 3 \cdot T(\bar{\mathbf{6}}) + 3 \cdot T(\mathbf{6}) = 3 \cdot 1 + 3 \cdot 1 = 6. \quad (3.7)$$

At infinite coupling the SU(6) flavor symmetry is realized as the SU(6) factor of the  $E_6 \supset SU(2) \times SU(6)$  maximal subgroup for which  $I_{\text{SU}(6) \hookrightarrow E_6} = 1$ , from which it follows by (D.5) that

$$k_{\text{SU}(6)\text{-strong}} = I_{\text{SU}(6) \hookrightarrow E_6} k_{E_6} = 6. \quad (3.8)$$

The agreement of (3.7) and (3.8) is a non-trivial check on our proposal.

**The U(1) symmetry.** At weak coupling, the half-hypermultiplets in the  $(\mathbf{3}, \bar{\mathbf{6}})$  of the  $SU(3) \times SU(6)$  combined gauge and flavor symmetries have charge +1 under the U(1) flavor group, while those in the  $(\bar{\mathbf{3}}, \mathbf{6})$  have charge  $-1$ . (These charge assignments just amount to a choice of the normalization of the flavor U(1) generator.) Thus, if this U(1) were weakly gauged, the matter multiplets would contribute to the coefficient of its beta function an amount proportional to the U(1) current algebra central charge

$$k_{\text{U}(1)\text{-weak}} = 3 \cdot 6 \cdot (+1)^2 + 3 \cdot 6 \cdot (-1)^2 = 36. \quad (3.9)$$

At infinite coupling the U(1) flavor symmetry is realized as the SO(2) rotation symmetry of the two half-hypermultiplets in the  $\mathbf{2}$  of the SU(2) gauge group. Then one  $\mathbf{2}$  has U(1) charge  $+q$  and the other has charge  $-q$ , but we can't determine  $q$  *a priori* since we don't have a direct way of comparing the normalization of the U(1) generator at infinite coupling and at weak coupling. The  $E_6$  SCFT “matter” is not charged under the U(1). Upon weakly gauging the U(1), the contribution to its beta function from the matter multiplets will therefore be

$$k_{\text{U}(1)\text{-strong}} = 2 \cdot (+q)^2 + 2 \cdot (-q)^2 = 4q^2. \quad (3.10)$$

Equating (3.9) and (3.10) then implies  $q = 3$ .

Although this value of the U(1) flavor charge at infinite coupling cannot be used as a consistency check of the duality, it does provide interesting evidence for the identification of the SU(2)-doublet half-hypermultiplets at infinite coupling as magnetic monopoles of the weak coupling SU(3) gauge group. For, SU(3) monopoles which are singlets under the SU(6) flavor symmetry also have charge  $\pm 3$  under the U(1) flavor group. To see this, dress a monopole state  $|M\rangle$  with the fermionic zero-modes of the 6 hypermultiplets in the  $\mathbf{3} \oplus \bar{\mathbf{3}}$  of the SU(3) gauge group. This can be done by going to a point on the Coulomb branch with  $v = 0$  but  $u \neq 0$  breaking  $SU(3) \rightarrow U(1)^2$  and leaving one color-component of each hypermultiplet massless. These then contribute 6 massless Dirac fermions, giving 12 real zero modes, charged under the U(1) coming from the  $SU(2) \rightarrow U(1)$  breaking in the dual description. Split these zero modes into 6 creation operators  $c_i^\dagger$ , and 6 annihilation operators  $c_i$ , and take  $|M\rangle$  to be the state annihilated by the  $c_i$ , so that  $|M\rangle$  is an SU(6) singlet. Note that the  $c_i^\dagger$  carry charge +1 under the U(1) flavor symmetry, corresponding

to the normalization we chose above (3.9). The spectrum of monopole states is then given by  $c_{i_1}^\dagger \cdots c_{i_n}^\dagger |M\rangle$  with  $n = 0, \dots, 6$ . If the  $U(1)$  charge of  $|M\rangle$  is  $-q$ , then these states will have  $U(1)$  charges  $n - q$ . The two  $SU(6)$  singlet states,  $|M\rangle$  and  $c_1^\dagger \cdots c_6^\dagger |M\rangle$ , therefore have charges  $-q$  and  $6 - q$ . CPT invariance implies these must be opposite, giving  $q = 3$ .

**R-symmetry central charges.** The central charges for the  $U(1)_R$  and  $SU(2)_R$  factors are proportional because their generators are both descendants of a single primary,  $T$ . It follows from normalizing them on, say, a free multiplet, as done in appendix D, that  $k_{U(1)_R} = 8k_{SU(2)_R}$ .

$T$  is the primary of a supermultiplet with  $R = I = j = \tilde{j} = 0$  and  $D = 2$ . The  $R$ -symmetry central charges are proportional to the  $\langle TT \rangle$  2-point function, and their derivative with respect to the marginal gauge coupling is proportional to  $\langle \Phi TT \rangle$  where  $\Phi$  is a chiral supermultiplet with  $R = 4$ , and so vanish by the same argument as in the last subsection. Therefore the  $R$ -symmetry central charges are also independent of the gauge coupling, so are the same at weak and infinite coupling.

From the  $U(1)_R$  free field central charge normalization given in appendix D, a short calculation gives  $k_{U(1)_R - SU(3)} = 136/3$  for the weakly coupled  $SU(3)$  theory.<sup>3</sup> Likewise, the weakly coupled  $SU(2)$  vector multiplet and two  $\mathbf{2}$  half-hypermultiplets contribute  $k_{U(1)_R - SU(2)} = 32/3$ . Taking their difference, we deduce that the  $E_6$  SCFT contributes

$$k_{U(1)_R - E_6} = \frac{104}{3}.$$

#### 4. Infinite coupling in $Sp(2)w/12 \cdot 4$

We now quickly run through the same reasoning described in detail in the last two sections to support the equivalence

$$Sp(2) w/ 12 \cdot 4 = SU(2) w/ SCFT_{E_7}, \tag{4.1}$$

of the (rank two) Lagrangian SCFT on the left with the strongly coupled SCFT on the right.

As reviewed in appendix A.2, the curve for the  $Sp(2)$  theory is

$$y^2 = x \left[ (1 - \sqrt{f})x^2 - ux - v \right] \left[ (1 + \sqrt{f})x^2 - ux - v \right], \tag{4.2}$$

where  $u$  and  $v$  are the Coulomb branch vevs of dimension 2 and 4, respectively, and their associated basis of holomorphic one-forms are given in (A.3). The infinite coupling point is at  $f = 1$ , where the curve degenerates to a genus one curve, and it is easy to check that the  $\omega_u = xdx/y$  one-form develops a pair of poles at  $x = \infty$ , whereas  $\omega_v = dx/y$  remains holomorphic. Thus, as in the  $SU(3)$  example, when  $f = 1$ ,  $v$  is the Coulomb branch vev of a rank 1 SCFT in which  $u$  appears as a mass deformation, while for  $f \sim 1$ ,  $u$  is the vev of an  $SU(2)$  vector multiplet weakly gauging some global symmetry of this rank 1 SCFT.

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<sup>3</sup>Note that this does not agree with the  $N = 1$   $U(1)_R$  central charge,  $\tau_{RR}$ , computed in [13], because the  $N = 1$  and  $N = 2$   $U(1)_R$  charges are not the same, but are related by  $R_{N=1} = \frac{1}{3}R_{N=2} - \frac{4}{3}I_3$ .

Isolating the curve of the  $SU(2)$  factor which weakly gauges part of the flavor symmetry of the  $E_7$  SCFT by setting  $v = 0$  in (4.2), and scaling to the non-singular genus one curve by defining  $\hat{y} = y/x$ , gives  $\hat{y}^2 = x[(x - u)^2 - fx^2]$  and  $\omega_u = dx/\hat{y}$ , which are precisely the curve and one-form [15] of the scale invariant  $Sp(1) N = 2$  superQCD.<sup>4</sup> It is weakly coupled at both  $f = 0$  and  $f = 1$  by virtue of the  $SL(2, \mathbb{Z})$  S-duality of this theory.

Likewise, isolating the curve of the rank 1 SCFT by setting  $f = 1$ , and making the change of variables

$$x = \frac{(\tilde{v} - \frac{1}{2}\tilde{u}^2)^2}{\tilde{x} - 2\tilde{u}(\tilde{v} - \frac{1}{2}\tilde{u}^2)}, \quad y = \frac{-(\tilde{v} - \frac{1}{2}\tilde{u}^2)^2 \tilde{y}}{(\tilde{x} - 2\tilde{u}(\tilde{v} - \frac{1}{2}\tilde{u}^2))^2}, \quad u = 3\tilde{u}, \quad v = \tilde{v} - \frac{1}{2}\tilde{u}^2, \quad (4.3)$$

we get

$$\tilde{y}^2 = \tilde{x}^3 - \left(2\tilde{v}^3 - \frac{3}{2}\tilde{v}\tilde{u}^4 + \frac{1}{2}\tilde{u}^6\right)\tilde{x} - \left(2\tilde{v}^4\tilde{u} - 2\tilde{v}^3\tilde{u}^3 + \frac{1}{2}\tilde{v}\tilde{u}^7 - \frac{1}{8}\tilde{u}^9\right), \quad \tilde{\omega}_{\tilde{v}} = \frac{d\tilde{x}}{\tilde{y}}. \quad (4.4)$$

When  $\tilde{u} = 0$ , this is the curve of the rank 1  $E_7$  SCFT, reviewed in appendix B.

With  $u \neq 0$ , this gives a mass deformation of the  $E_7$  SCFT curve. Comparing to the general mass deformation of the  $E_7$  SCFT (B.7), gives the  $E_7$  adjoint casimirs

$$M_2 = 2u, \quad M_6 = -2u^3, \quad M_8 = -\frac{3}{2}u^4, \quad M_{10} = 0, \quad M_{12} = \frac{1}{2}u^6, \quad M_{14} = \frac{1}{2}u^7, \quad M_{18} = -\frac{1}{8}u^9. \quad (4.5)$$

The  $M_n$  are given [5] in terms of the eigenvalues  $m_i$  ( $i = 1, \dots, 6$ ) and  $m$  of the Cartan subalgebra of the  $SO(12) \times SU(2)$  maximal subgroup of  $E_7$  in appendix B.2. A solution of (4.5) is given by  $m = \sqrt{6\tilde{u}}$  and  $m_i = 0$ , showing that the  $\tilde{u}$  vev breaks  $SU(2) \times SO(12) \rightarrow U(1) \times SO(12)$ , thus identifying the  $SO(12)$  flavor symmetry expected from an  $Sp(2)$  theory with six massless fundamental hypermultiplets.<sup>5</sup>

The vanishing of the beta function for the  $SU(2)$  gauge coupling implies, as argued in section 3, that the central charge of the SCFT flavor current algebra be given by (3.5). Applying this to the present example using group theory data from appendix C, we compute the central charge of the  $E_7$  SCFT current algebra to be

$$k_{E_7} = 2 \cdot T(\mathbf{3})/I_{SU(2) \hookrightarrow E_7} = 8. \quad (4.6)$$

<sup>4</sup>The change of variables showing its equivalence to the  $SU(2)$  form of the curve (2.6) is discussed in [15].

<sup>5</sup>As in the footnote in section 2.2, some algebra shows that (4.5) is actually consistent with three different mass assignments (up to permutations),

$$\begin{aligned} \text{(a):} & \quad m = \sqrt{6\tilde{u}}, \quad m_{1,\dots,6} = 0, \quad \iff \mathbf{su}(2) \times SO(12) \rightarrow \mathbf{u}(1) \times SO(12), \\ \text{(b):} & \quad m = m_{1,\dots,4} = 0, \quad m_5 = m_6 = \sqrt{3\tilde{u}/2}, \quad \iff \mathbf{su}(2) \times SO(12) \rightarrow \mathbf{su}(2) \times SO(8) \times SU(2) \times U(1), \\ \text{(c):} & \quad m = \sqrt{3\tilde{u}/2}, \quad m_{1,2,3,4,5} = -m_6 = \sqrt{3\tilde{u}/8}, \quad \iff \mathbf{su}(2) \times SO(12) \rightarrow \mathbf{u}(1) \times SU(6) \times SU(2), \end{aligned}$$

where the corresponding adjoint breaking patterns of the  $SU(2) \times SO(12)$  maximal subgroup of  $E_7$  are shown on the right. The (a) breaking, which manifestly leaves an unbroken  $SO(12)$  factor, is the one described above. The (b) breaking pattern actually also leaves an  $SO(12)$  unbroken. This is not manifest because the unbroken  $SO(12)$  does not coincide with the  $SO(12)$  used for the basis of Casimirs. The (c) breaking pattern is different, corresponding to adjoint breaking of the  $SU(3)$  factor in the  $SU(3) \times SU(6) \subset E_7$  maximal subgroup, and does not give the expected global symmetry group of the original  $Sp(2)$  SCFT.



This can be independently checked by comparing the  $SO(12)$  flavor algebra central charge in the weak coupling and infinite coupling descriptions. At weak coupling the half-hypermultiplets transform as a  $(\mathbf{4}, \mathbf{12})$  under the  $Sp(2) \times SO(12)$  combined gauge and flavor symmetries. Thus the central charge of the  $SO(12)$  flavor current is by (D.6)

$$k_{SO(12)\text{-weak}} = 4 \cdot T(\mathbf{12}) = 8. \tag{4.7}$$

At infinite coupling it follows by (D.5) that

$$k_{SO(12)\text{-strong}} = I_{SO(12) \hookrightarrow E_7} k_{E_7} = 8. \tag{4.8}$$

The agreement of (4.7) and (4.8) is a non-trivial check of the duality.

Finally, by comparing  $U(1)_R$  central charges in the two dual theories, we deduce that the  $E_7$  SCFT contributes

$$k_{U(1)_R\text{-}E_7} = k_{U(1)_R\text{-}Sp(2)} - k_{U(1)_R\text{-}SU(2)} = \frac{176}{3} - 8 = \frac{152}{3}.$$

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## A. Rank 2 lagrangian SCFTs

Here we briefly review the systematics of Lagrangians for  $N = 2$  superQCD, and then collect from the literature the known curves for the rank 2 scale-invariant superQCDs.

### A.1 Scale-invariant rank 2 superQCDs

It is a straightforward group theory exercise to determine all the rank two  $N = 2$  theories with vanishing beta function. They are listed in table 1, along with their flavor symmetries, and whether or not they have infinite coupling points in their spaces of couplings.

The table does not include the three theories which are products of two decoupled rank 1 scale-invariant gauge theories. These rank 1 theories are the  $SU(2)$  theories with either two adjoint or eight fundamental half-hypermultiplets. The first is the  $N = 4$   $SU(2)$  theory, and both were examined in [2]. Both have an  $SL(2, \mathbb{Z})$  S-duality group, and so only have weakly coupled limits.

Theories #1 and #2 in table 1 have two marginal couplings. Their low energy effective actions were found in [11]. The S-duality group of theory #2 was determined in [9, 10], and the self-duality of the  $SU(2)$  factors were found to eliminate any infinite coupling limits. A similar analysis has not been performed for theory #1, but it seems probable that it has no infinite coupling limits, for the same reason. Indeed, taking the coupling of one of the  $SU(2)$  factors small, the  $SL(2, \mathbb{Z})$  duality of the other factor eliminates any infinite-coupling

#	Gauge group	Half-hypermultiplets	Flavor symmetry	Infinite coupling?
1	$SU(2) \times SU(2)$	$4 \cdot (\mathbf{2}, \mathbf{2})$	$Sp(2)$	no?
2	$SU(2) \times SU(2)$	$2 \cdot ((\mathbf{2}, \mathbf{2}) \oplus (\mathbf{2}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{2}))$	$Sp(1) \times SO(2)^2$	no
3	$SU(3)$	$2 \cdot \mathbf{8}$	$Sp(1)$	no
4	$SU(3)$	$6 \cdot (\mathbf{3} \oplus \bar{\mathbf{3}})$	$U(6)$	yes
5	$SU(3)$	$\mathbf{3} \oplus \bar{\mathbf{3}} \oplus \mathbf{6} \oplus \bar{\mathbf{6}}$	$U(1)^2$	yes
6	$Sp(2)$	$2 \cdot \mathbf{10}$	$Sp(1)$	no
7	$Sp(2)$	$12 \cdot \mathbf{4}$	$SO(12)$	yes
8	$Sp(2)$	$8 \cdot \mathbf{4} \oplus 2 \cdot \mathbf{5}$	$SO(8) \times Sp(1)$	no
9	$Sp(2)$	$4 \cdot \mathbf{4} \oplus 4 \cdot \mathbf{5}$	$SO(4) \times Sp(2)$	?
10	$Sp(2)$	$6 \cdot \mathbf{5}$	$Sp(3)$	yes
11	$G_2$	$2 \cdot \mathbf{14}$	$Sp(1)$	no
12	$G_2$	$8 \cdot \mathbf{7}$	$Sp(4)$	yes

**Table 1:** Rank 2 scale-invariant  $N = 2$  gauge theories which are not products of two rank 1 theories.

limit in its coupling. Continuing this to strong coupling in the first factor then disallows any one-(complex-)dimensional submanifolds of infinite coupling. However, the possibility remains of an isolated infinite coupling point at strong coupling in both factors.

Theories #3, #6, and #11 are  $N = 4$  theories, all of which are self-dual [1]. (Note that  $Sp(2) \simeq SO(5)$ , so that the  $Sp(2)$  theory is actually self-dual.)

The low energy effective action of theory #8 was determined in [7, 8] and indicates an  $SL(2, \mathbb{Z})$  duality group with only weakly coupled limits. The low energy effective action of #9, and whether it has an infinite coupling limit, is not known.

The effective actions of the remaining theories imply that they have infinite coupling limits, as shown for #4 in [3], #5 in [16], #7 and #10 in [15], and #12 in [17]. We now describe their effective actions in more detail.

### A.2 Rank 2 effective actions

The effective actions on the Coulomb branches of the above-mentioned theories are most conveniently encoded in the associated genus 2 curve. Recall [2] (see, e.g., [17] for a brief review) that the low energy effective action on the Coulomb branch of a theory with rank  $r$  gauge group  $G$  is an  $N = 2$   $U(1)^r$  theory, parametrized by the  $r$  vevs of the complex scalars in each  $U(1)$  vector multiplet. Taking  $\{u^k\}$  as  $r$  good complex coordinates on the Coulomb branch, the matrix  $\tau_{ij}(u^k)$  of complex  $U(1)^r$  couplings and the central charge  $Z(u^k)$  of the  $N = 2$  algebra can be encoded (at least for  $r < 4$ ) by a holomorphic family of genus  $r$  Riemann surfaces  $\Sigma(u^k)$  together with a specified basis  $\{\omega_i\}$  of the  $r$  holomorphic one-forms on  $\Sigma$ :

$$\tau_{ij} = A_{ik}(B^{-1})^k_j, \quad \frac{\partial Z(\gamma)}{\partial u^k} = \oint_{\gamma} \omega_k, \quad (\text{A.1})$$

where  $A_{ik} := \oint_{\alpha_i} \omega_k$ ,  $B_k^j := \oint_{\beta_j} \omega_k$ ,  $\{\alpha_i, \beta^j\}$  are a basis of homology one forms with canonical intersection matrix, and the homology class of the contour  $\gamma$  is determined by

the electric and magnetic charges of the state. Note in particular, that under holomorphic changes of variables on the Coulomb branch  $u^k \rightarrow \tilde{u}^k$ , the holomorphic one forms transform as  $\omega_k \rightarrow \tilde{\omega}_k = (\partial u^\ell / \partial \tilde{u}^k) \omega_\ell$ , since it is the central charge that remains invariant.

Turning on masses  $m_a$  in these theories corresponds to deformations of the Riemann surfaces  $\Sigma(u^k, m_a)$  such that there exists a central charge  $Z$  which depends linearly on the masses with integer coefficients [2]. This means that the second equation in (A.1) can be integrated to  $Z(\gamma) = \oint_\gamma \lambda$ , where  $\lambda$  is a meromorphic one-form whose residues are integral linear combinations of the  $m_a$ , and which satisfies

$$\frac{\partial \lambda}{\partial u^k} = \omega_k + df_k, \tag{A.2}$$

where  $df_k$  are total derivatives on the curve  $\Sigma$ .

Specializing to rank 2, call the two coordinates on the Coulomb branch  $u$  and  $v$ . Since all genus 2 Riemann surfaces are hyperelliptic, they can all be described as complex curves in a 2-dimensional projective space of the form  $y^2 = \mathcal{P}(x)$  where  $\mathcal{P}$  is a fifth- or sixth-order polynomial in  $x$ . This realizes the Riemann surface as a 2-sheeted cover of the complex  $x$ -plane (plus the point at infinity) branched at six points (the zeros of  $\mathcal{P}$ ).  $\mathcal{P}$  can vary holomorphically with  $u$  and  $v$ , and degenerations of the curve correspond to collisions of the branch points. It is possible, by suitable coordinate changes, to choose the basis of holomorphic one forms to be

$$\omega_u = \frac{x dx}{y}, \quad \omega_v = \frac{dx}{y}. \tag{A.3}$$

We will use such coordinates in what follows.

**SU(3) $w/6 \cdot (3 \oplus \bar{3})$  and SU(3) $w/3 \oplus \bar{3} \oplus 6 \oplus \bar{6}$ .** The curve for the scale-invariant theory with 6 fundamental hypermultiplets is [3, 15]

$$y^2 = (x^3 - ux - v)^2 - fx^6 \tag{A.4}$$

where  $f$  is a holomorphic function of the microscopic gauge coupling  $\tau$ . At weak coupling  $f \sim e^{2\pi i \tau}$ . The curve degenerates whenever the discriminant in  $x$  of its right side vanishes. For (A.4) the discriminant is  $f^3(f - 1)$  times factors that depend on  $u$  and  $v$  moduli. The vanishing of the moduli-dependent factors determines submanifolds on the Coulomb branch where various dyons become massless. The coupling-dependent prefactor, on the other hand, indicates values of the couplings where there are singularities in the effective action everywhere on the Coulomb branch: the curve becomes singular at  $f = 0$  and  $f = 1$ , irrespective of the values of the Coulomb branch vevs  $u, v$ . The  $f = 0$  singularity has the interpretation as the weak coupling limit of the SU(3) gauge theory, while the  $f = 1$  singularity is the infinite coupling singularity which is the subject of this paper.

The curve for the scale-invariant theory with one symmetric and one antisymmetric hypermultiplet is also given by (A.4). This can be deduced from [16]. These are nevertheless different theories. In particular they have different global symmetry groups and different mass deformations. This gives an example of distinct scale invariant theories with the same scale-invariant form, but different mass deformations.

The 6-flavor theory has a  $U(1) \times SU(6)$  global flavor symmetry group, and therefore a deformation by 6 mass parameters: the  $U(1)$  mass  $M$  of dimension 1, and the five adjoint Casimirs of  $SU(6)$ ,  $S_n$ , with dimension  $n$ , for  $n = 2, \dots, 6$ . The explicit mass deformations of (A.4) for the 6 flavor theory is given in [3, 15]:

$$y^2 = \left[ (x + \sqrt{1 - fM})^3 - u(x + \sqrt{1 - fM}) - v \right]^2 - f [x^6 - S_2x^4 - S_3x^3 - S_4x^2 - S_5x - S_6]. \quad (\text{A.5})$$

If the  $m_i$  are the eigenvalues of the  $SU(6)$  adjoint mass matrix satisfying  $\sum_i m_i = 0$ , then the  $S_n$  Casimirs are given by  $S_n := \sum_{i_1 < \dots < i_n} m_{i_1} \cdots m_{i_n}$ . With this specific mass dependence of the curve, one can then integrate (A.2) to find  $\lambda$  and thus a central charge  $Z(u, v; M, m_i)$  which depends linearly on the mass eigenvalues with integer coefficients. It is worth emphasizing that the normalization of the masses and the specific basis of the flavor symmetry adjoint Casimirs,  $S_n$ , that enter into (A.5) are determined by linear integral dependence of the central charge on the masses.

The symmetric plus antisymmetric theory has global flavor symmetry  $U(1) \times U(1)$  and two mass deformation parameters both of dimension 1. The mass deformation of (A.4) for the symmetric plus antisymmetric theory is not known, though the deformation of the double-cover curve is given in [16].

**Sp(2) $w$ /12 · 4 and Sp(2) $w$ /6 · 5.** The scale-invariant curve is, in either case, [15]

$$y^2 = x(x^2 - ux - v)^2 - fx^5, \quad (\text{A.6})$$

and degenerates for all  $u$  and  $v$  whenever the coupling  $f = 0$  or  $f = 1$ . The  $f = 0$  singularity is the weak coupling limit of the  $Sp(2)$  gauge theory, while the  $f = 1$  singularity is the new infinite coupling limit.  $u$  and  $v$  are Coulomb branch vevs of dimension 2 and 4, respectively. This is another example of the same scale-invariant curve having two inequivalent mass deformations.

The theory with 12 half-hypermultiplets in the 4-dimensional representation has global flavor symmetry group  $SO(12)$ , and therefore 6 mass parameters with dimensions 2, 4, 6, 6, 8, and 10. This mass deformations of (A.6) is [15]

$$y^2 = x(x - u)^2 - 2\sqrt{f}(x - u)s_6 - f(x^5 - S_2x^4 + S_4x^3 - S_6x^2 + S_8x - S_{10}). \quad (\text{A.7})$$

If  $\pm m_i$ ,  $i = 1, \dots, 6$  are the eigenvalues of the  $SO(12)$  adjoint mass matrix, then the mass parameters appearing in the curve are the Casimirs  $S_{2n} := \sum_{i_1 < \dots < i_n} m_{i_1}^2 \cdots m_{i_n}^2$ , and  $s_6 := \prod_i m_i$ .

The theory with 6 half-hypermultiplets in the 5-dimensional representation has global flavor symmetry group  $Sp(3)$  and 3 mass parameters of dimensions 2, 4, and 6. This mass deformation of (A.6) is also given in [15].

**G<sub>2</sub> w/ 8 · 7.** The curve for the scale invariant theory is [17]

$$vy^2 = (x^3 - uvx - 2v^2)^2 - fx^6, \quad (\text{A.8})$$

and again has a weak coupling singularity at  $f = 0$  and an infinite coupling singularity at  $f = 1$ .  $u$  and  $v$  are Coulomb branch vevs of dimension 2 and 6. The global flavor symmetry

of this theory is  $\text{Sp}(4)$ , and so the curve should have a 4-parameter mass deformation with masses of dimension 2, 4, 6, and 8; however the explicit form of this deformation is not known.

## B. The $E_6$ and $E_7$ rank 1 SCFTs

The curves and one-forms encoding the effective action on the Coulomb branch for the rank 1  $N = 2$  SCFTs with  $E_n$  global symmetry groups was first worked out in [5]. Since these are all rank one theories, the curves are elliptic (genus 1 Riemann surfaces) of the form  $\tilde{y}^2 = \tilde{x}^3 + \dots$ , and we choose the basis of holomorphic one forms to be  $\omega = d\tilde{x}/\tilde{y}$ .

### B.1 $E_6$

The curve for the scale-invariant  $E_6$  SCFT (i.e., without mass deformations) is

$$\tilde{y}^2 = \tilde{x}^3 - \tilde{v}^4. \tag{B.1}$$

The Coulomb branch vev,  $\tilde{v}$ , has mass dimension 3.

The maximal mass deformation of this curve is

$$\tilde{y}^2 = \tilde{x}^3 - (M_2\tilde{v}^2 + M_5\tilde{v} + M_8)\tilde{x} - (\tilde{v}^4 + M_6\tilde{v}^2 + M_9\tilde{v} + M_{12}). \tag{B.2}$$

Here we have added all possible terms which deform the complex structure of the curve. Terms proportional to  $\tilde{x}^2$  do not appear since they can be reabsorbed in a shift in the  $\tilde{x}$  variable. Likewise, an  $M_3\tilde{v}^3$  term does not appear since its deformation is simply a shift in the Coulomb branch vev  $\tilde{v}$ . The subscripts of the remaining six mass parameters record their mass dimensions. They are the dimensions of the adjoint Casimirs of  $E_6$ , hinting that they break an  $E_6$  global symmetry group.

To confirm this, one must construct a central charge  $Z$  which depends linearly on the  $E_6$  mass eigenvalues with integer coefficients [2]. Calling the six mass eigenvalues of the  $E_6$  mass matrix  $m_a$ ,  $a = 1, \dots, 6$ , then the integers  $n^a = \partial Z(\gamma)/\partial m_a$  are the ‘‘quark number’’ charges of the generically unbroken  $U(1)^6$  flavor symmetry. Thus one must find a specific  $E_6$ -Weyl-invariant polynomial form for the Casimirs in terms of the eigenvalues,  $M_n(m_a)$ , such that the second equation in (A.1) can be integrated to  $Z(\gamma) = \oint_\gamma \lambda$ , where  $\lambda$  is a meromorphic one-form whose residues are integral linear combinations of the  $m_a$ , and satisfying  $\partial\lambda/\partial\tilde{u} = \omega + df$  where  $df$  is a total derivative on the curve. Following a method described in section 17 of [2], this was done for the  $E_n$  curves in [5, 18].

The result of [5] for the  $E_6$  mass deformation is

$$\begin{aligned} M_2 &= -\frac{1}{3}P_2, & M_5 &= \frac{2}{3}P_5, & M_6 &= \frac{2}{3}P_6 - \frac{7}{108}P_2^3 \\ M_8 &= -\frac{7}{432}P_2^4 + \frac{11}{45}P_2P_6 - \frac{8}{15}P_8, \\ M_9 &= \frac{1}{18}P_2^2P_5 - \frac{8}{21}P_9 \\ M_{12} &= \frac{32}{135}P_{12} - \frac{298}{18225}P_2^2P_8 - \frac{101}{218700}P_2^3P_6 + \frac{13}{405}P_6^2 - \frac{49}{1049760}P_2^6 - \frac{19}{3645}P_2P_5^2, \end{aligned} \tag{B.3}$$

where the  $P_n$  are the basis of  $E_6$  Casimirs given by [19]. These Casimirs can be written in terms of the Casimirs  $\tilde{S}_n$  and  $\tilde{T}$  of the  $SU(6) \times SU(2)$  maximal subgroup of  $E_6$  as<sup>6</sup>

$$\begin{aligned}
 P_2 &= -6\tilde{T} + 6\tilde{S}_2, \\
 P_5 &= 12\tilde{T}\tilde{S}_3 + 12\tilde{S}_5, \\
 P_6 &= -20\tilde{T}^3 + 64\tilde{T}^2\tilde{S}_2 - 64\tilde{T}\tilde{S}_2^2 + 20\tilde{S}_2^3 - 3\tilde{S}_3^2 + 20\tilde{T}\tilde{S}_4 + 4\tilde{S}_2\tilde{S}_4 - 24\tilde{S}_6, \\
 P_8 &= 15\tilde{T}^4 - 76\tilde{T}^3\tilde{S}_2 + 112\tilde{T}^2\tilde{S}_2^2 - 76\tilde{T}\tilde{S}_2^3 + 15\tilde{S}_2^4 - 3\tilde{T}\tilde{S}_3^2 - 12\tilde{S}_2\tilde{S}_3^2 - 50\tilde{T}^2\tilde{S}_4 \\
 &\quad + 54\tilde{T}\tilde{S}_2\tilde{S}_4 + 16\tilde{S}_2^2\tilde{S}_4 - 10\tilde{S}_4^2 + 15\tilde{S}_3\tilde{S}_5 + 186\tilde{T}\tilde{S}_6 - 66\tilde{S}_2\tilde{S}_6, \\
 P_9 &= 56\tilde{T}^3\tilde{S}_3 - 140\tilde{T}^2\tilde{S}_2\tilde{S}_3 + 56\tilde{T}\tilde{S}_2^2\tilde{S}_3 - 56\tilde{T}\tilde{S}_3\tilde{S}_4 + 140\tilde{T}^2\tilde{S}_5 - 56\tilde{T}\tilde{S}_2\tilde{S}_5 \\
 &\quad + 56\tilde{S}_2^2\tilde{S}_5 + 28\tilde{S}_4\tilde{S}_5 - 84\tilde{S}_3\tilde{S}_6, \\
 P_{12} &= \tilde{T}^6 - 22\tilde{T}^5\tilde{S}_2 + 67\tilde{T}^4\tilde{S}_2^2 - 72\tilde{T}^3\tilde{S}_2^3 + 67\tilde{T}^2\tilde{S}_2^4 - 22\tilde{T}\tilde{S}_2^5 + \tilde{S}_2^6 - 33\tilde{T}^3\tilde{S}_3^2 \\
 &\quad + 72\tilde{T}^2\tilde{S}_2\tilde{S}_3^2 + 18\tilde{T}\tilde{S}_2^2\tilde{S}_3^2 - 12\tilde{S}_2^3\tilde{S}_3^2 + 3\tilde{S}_3^4 - 28\tilde{T}^4\tilde{S}_4 + 66\tilde{T}^3\tilde{S}_2\tilde{S}_4 \\
 &\quad - 118\tilde{T}^2\tilde{S}_2^2\tilde{S}_4 + 4\tilde{T}\tilde{S}_2^3\tilde{S}_4 + 16\tilde{S}_2^4\tilde{S}_4 + 50\tilde{T}\tilde{S}_3^2\tilde{S}_4 - 8\tilde{S}_2\tilde{S}_3^2\tilde{S}_4 - 64\tilde{T}^2\tilde{S}_4^2 \\
 &\quad + 58\tilde{T}\tilde{S}_2\tilde{S}_4^2 - 12\tilde{S}_2^2\tilde{S}_4^2 - 20\tilde{S}_4^3 - 147\tilde{T}^2\tilde{S}_3\tilde{S}_5 - 194\tilde{T}\tilde{S}_2\tilde{S}_3\tilde{S}_5 + 26\tilde{S}_2^2\tilde{S}_3\tilde{S}_5 \\
 &\quad + 45\tilde{S}_3\tilde{S}_4\tilde{S}_5 + 251\tilde{T}\tilde{S}_5^2 + 19\tilde{S}_2\tilde{S}_5^2 + 522\tilde{T}^3\tilde{S}_6 - 342\tilde{T}^2\tilde{S}_2\tilde{S}_6 + 584\tilde{T}\tilde{S}_2^2\tilde{S}_6 \\
 &\quad - 44\tilde{S}_2^3\tilde{S}_6 - 87\tilde{S}_3^2\tilde{S}_6 - 590\tilde{T}\tilde{S}_4\tilde{S}_6 + 26\tilde{S}_2\tilde{S}_4\tilde{S}_6 - 78\tilde{S}_6^2.
 \end{aligned} \tag{B.4}$$

We have defined the  $\tilde{S}_n$  and  $\tilde{T}$  Casimirs by

$$\tilde{S}_n := \sum_{i_1 < \dots < i_n} m_{i_1} \dots m_{i_n} \quad (n = 2, \dots, 6), \quad \tilde{T} := m^2, \tag{B.5}$$

where  $\pm m$  are the mass eigenvalues in the  $SU(2)$  factor, and  $m_i$  for  $i = 1, \dots, 6$  with  $\sum_i m_i = 0$  are the mass eigenvalues for the Cartan subalgebra of the  $SU(6)$  factor. The associated meromorphic one-form is given in [5], but will not be needed here.

## B.2 $E_7$

The curve for the scale-invariant  $E_7$  SCFT is

$$y^2 = x^3 - 2u^3x. \tag{B.6}$$

The Coulomb branch vev  $u$  has mass dimension 4. The factor of 2 is a normalization (of  $u$ ) chosen to match to that of [5]. The maximal mass deformation of this curve is

$$y^2 = x^3 - (2u^3 + M_8u + M_{12})x - (M_2u^4 + M_6u^3 + M_{10}u^2 + M_{14}u + M_{18}). \tag{B.7}$$

The deformation parameters  $M_n$  are adjoint Casimirs of  $E_7$  determined in [5] to be the following expressions<sup>7</sup> in terms of the Casimirs  $T_n$ ,  $t_6$ , and  $U$  of the  $SO(12) \times SU(2)$  maximal

<sup>6</sup>In [19, 5] the  $E_6$  Casimirs were expressed in terms of a basis of Casimirs of the  $SO(10) \times U(1)$  maximal subgroup of  $E_6$ . For our purposes it is much more convenient to express them in terms of a basis of  $SU(6) \times SU(2)$  Casimirs. To do that, we followed the method of [19] by computing the character of the **27** of  $E_6$  using the fact that under  $SU(6) \times SU(2) \subset E_6$ , the **27** decomposes as  $\mathbf{27} = (\mathbf{6}, \mathbf{2}) \oplus (\overline{\mathbf{15}}, \mathbf{1})$ .

<sup>7</sup>We have shifted our Coulomb vev  $u$  relative to that of [5] to eliminate a  $u^2x$  term, and consequently also shifted their  $P_n$  Casimirs to the  $M_n$  values shown in (B.8).

subgroup of  $E_7$ :

$$\begin{aligned}
M_2 &= P_2, & M_6 &= P_6 - \frac{2}{3}P_2P_4, & M_8 &= P_8 - \frac{1}{6}P_4^2, \\
M_{10} &= P_{10} - \frac{1}{2}P_4P_6 + \frac{1}{6}P_2P_4^2, \\
M_{12} &= P_{12} - \frac{1}{6}P_4P_8 + \frac{1}{54}P_4^3, \\
M_{14} &= P_{14} - \frac{1}{3}P_4P_{10} + \frac{1}{12}P_4^2P_6 - \frac{1}{54}P_2P_4^3, \\
M_{18} &= P_{18} - \frac{1}{6}P_4P_{14} + \frac{1}{36}P_4^2P_{10} - \frac{1}{216}P_4^3P_6 + \frac{1}{1296}P_2P_4^4,
\end{aligned} \tag{B.8}$$

where

$$\begin{aligned}
P_2 &= U + \frac{2}{3}\tilde{T}_2, & P_4 &= \frac{1}{12}\tilde{T}_2^2 + T_4, \\
P_6 &= -\frac{1}{108}\tilde{T}_2^3 + \frac{1}{3}\tilde{T}_2T_4 + \frac{20}{3}t_6 + \frac{2}{3}T_6, & P_8 &= 8Ut_6 + \frac{2}{3}\tilde{T}_2t_6 + \frac{2}{3}\tilde{T}_2T_6 - 2T_8, \\
P_{10} &= 4T_{10} - \frac{10}{3}U\tilde{T}_2t_6 - \frac{29}{18}\tilde{T}_2^2t_6 + \frac{22}{3}T_4t_6 + \frac{5}{36}\tilde{T}_2^2T_6 + \frac{1}{3}T_4T_6 - 2UT_8 - \frac{2}{3}\tilde{T}_2T_8, \\
P_{12} &= 4UT_{10} - \frac{1}{4}\tilde{T}_2^3t_6 + \tilde{T}_2T_4t_6 + \frac{4}{3}t_6^2 - \frac{4}{3}t_6T_6 + \frac{1}{3}T_6^2 + \frac{1}{4}\tilde{T}_2^2T_8 - T_4T_8, \\
P_{14} &= -\frac{8}{3}U\tilde{T}_2T_{10} - \tilde{T}_2^2T_{10} + 4T_{10}T_4 + \frac{1}{24}\tilde{T}_2^4t_6 - \frac{2}{3}\tilde{T}_2^2T_4t_6 + 2T_4^2t_6 + \frac{32}{3}Ut_6^2 - \frac{4}{9}\tilde{T}_2t_6^2 \\
&\quad - \frac{16}{3}Ut_6T_6 - \frac{2}{9}\tilde{T}_2t_6T_6 + \frac{2}{9}\tilde{T}_2T_6^2 + \frac{1}{12}\tilde{T}_2^3T_8 - \frac{1}{3}\tilde{T}_2T_4T_8 + \frac{4}{3}t_6T_8 - \frac{2}{3}T_6T_8, \\
P_{18} &= \frac{1}{16}\tilde{T}_2^4T_{10} - \frac{1}{2}\tilde{T}_2^2T_{10}T_4 + T_{10}T_4^2 + \frac{16}{3}UT_{10}t_6 + U\tilde{T}_2^2t_6^2 + \frac{1}{6}\tilde{T}_2^3t_6^2 - \frac{2}{3}\tilde{T}_2T_4t_6^2 - \frac{16}{27}t_6^3 \\
&\quad - \frac{8}{3}UT_{10}T_6 - \frac{1}{12}\tilde{T}_2^3t_6T_6 + \frac{1}{3}\tilde{T}_2T_4t_6T_6 + \frac{8}{9}t_6^2T_6 - \frac{4}{9}t_6T_6^2 + \frac{2}{27}T_6^3 - 2U\tilde{T}_2t_6T_8 \\
&\quad - \frac{1}{6}\tilde{T}_2^2t_6T_8 + \frac{2}{3}T_4t_6T_8 + \frac{1}{12}\tilde{T}_2^2T_6T_8 - \frac{1}{3}T_4T_6T_8 + UT_8^2.
\end{aligned} \tag{B.9}$$

Here  $\tilde{T}_2 := T_2 - U$  and the  $T_n$ ,  $t_6$ , and  $U$  Casimirs are defined by

$$T_{2n} := \sum_{i_1 < \dots < i_n} m_{i_1}^2 \dots m_{i_n}^2 \quad (n = 1, \dots, 5), \quad t_6 := \prod_i m_i, \quad U := m^2, \tag{B.10}$$

where  $\pm m$  are the mass eigenvalues in the  $SU(2)$  factor, and  $\pm m_i$  for  $i = 1, \dots, 6$  are the mass eigenvalues for the Cartan subalgebra of the  $SO(12)$  factor.

### C. Lie algebra indices

We normalize the inner product on the root space of simple Lie algebras by choosing the long roots to have length  $\sqrt{2}$ . This is the normalization in which the quadratic index of the  $\mathfrak{n}$  of  $SU(n)$  is  $T(\mathfrak{n}) = 1$ .

If under an embedding  $G \subset H$  of Lie algebras, the generators  $\{\tau^\alpha\}$  of  $G$  are related to the generators  $\{t^a\}$  of  $H$  by  $\tau^\alpha = \sum_a c_a^\alpha t^a$ , and a representation  $\mathfrak{r}$  of  $H$  decomposes as

$\oplus_i \mathbf{r}_i$  under  $G$ , then the Dynkin index of embedding of a Lie algebra  $G$  in  $H$  is

$$I_{G \hookrightarrow H} := \frac{\sum_{\alpha, a} (c_a^\alpha)^2}{\dim(G)} = \frac{\sum_i T(\mathbf{r}_i)}{T(\mathbf{r})}, \quad (\text{C.1})$$

independent of the choice of  $\mathbf{r}$ .

The examples in the body of the paper all turn out to give Dynkin index 1: From tables, e.g. [20], the  $\mathbf{27}$  of  $E_6$  decomposes as  $\mathbf{27} = (\mathbf{2}, \mathbf{6}) \oplus (\mathbf{1}, \mathbf{15})$  under the maximal subalgebra  $E_6 \supset \text{SU}(2) \times \text{SU}(6)$ . Thus for these embeddings

$$I_{\text{SU}(2) \hookrightarrow E_6} = \frac{6 \cdot T(\mathbf{2}) + 15 \cdot T(\mathbf{1})}{T(\mathbf{27})} = 1, \quad I_{\text{SU}(6) \hookrightarrow E_6} = \frac{2 \cdot T(\mathbf{6}) + 1 \cdot T(\mathbf{15})}{T(\mathbf{27})} = 1. \quad (\text{C.2})$$

Similarly,  $\mathbf{56} = (\mathbf{2}, \mathbf{12}) \oplus (\mathbf{1}, \mathbf{32})$  under  $E_7 \supset \text{SU}(2) \times \text{SO}(12)$ , giving

$$I_{\text{SU}(2) \hookrightarrow E_7} = \frac{12 \cdot T(\mathbf{2}) + 32 \cdot T(\mathbf{1})}{T(\mathbf{56})} = 1, \quad I_{\text{SO}(12) \hookrightarrow E_7} = \frac{2 \cdot T(\mathbf{12}) + 1 \cdot T(\mathbf{32})}{T(\mathbf{56})} = 1. \quad (\text{C.3})$$

## D. Normalization of central charges

From [13], the  $\text{U}(1)_i$ - $\text{U}(1)_j$  current 2-point function for free fields is given by

$$\langle J_\mu^i(x) J_\nu^j(0) \rangle = \left( \sum_b q_b^i q_b^j + 2 \sum_f q_f^i q_f^j \right) \frac{1}{4\pi^4} \frac{x^2 g_{\mu\nu} - 2x_\mu x_\nu}{x^8} \quad (\text{D.1})$$

where  $b$  runs over complex scalars and  $f$  over Weyl fermions, and  $q_{b,f}^i$  are their charges under the  $\text{U}(1)_i$  group, namely,  $[Q^i, \phi_b] = q_b^i \phi_b$  and  $[Q^i, \psi_f] = q_f^i \psi_f$ . The  $\text{U}(1)_i$  charges are related to the currents in the usual way by  $Q^i = \int d^3x J_0^i$ .

For  $n$  half-hypermultiplets of charges  $q_h^i = \delta_h^i$ ,  $h = 1, \dots, n$ , so that  $Q^i$  counts the number of  $i$ th half-hypermultiplets, this then gives

$$\langle J_\mu^i(x) J_\nu^j(0) \rangle = \frac{3\delta^{ij}}{4\pi^4} \frac{x^2 g_{\mu\nu} - 2x_\mu x_\nu}{x^8}. \quad (\text{D.2})$$

These  $n$   $\text{U}(1)$ 's form the Cartan subalgebra of the  $\text{U}(n)$  flavor symmetry rotating the  $n$  half-hypermultiplets. A basis of the  $n^2$  Hermitian  $\text{U}(n)$  generators in the fundamental are  $Q^{(ij)}$  and  $Q^{[ij]}$  with matrix elements  $[Q^{(ij)}]_{k\ell} = \frac{1}{2}(\delta_k^i \delta_\ell^j + \delta_\ell^i \delta_k^j)$  and  $[Q^{[ij]}]_{k\ell} = \frac{i}{2}(\delta_k^i \delta_\ell^j - \delta_\ell^i \delta_k^j)$ . Thus the  $\text{U}(1)_i$  generator  $Q^i = Q^{(ii)}$  (including normalization). Furthermore, it is easy to check that this is an orthogonal basis:

$$\text{tr}(Q^{(ij)} Q^{(k\ell)}) = \delta^{(ij)(k\ell)}, \quad \text{tr}(Q^{(ij)} Q^{[k\ell]}) = 0, \quad \text{tr}(Q^{[ij]} Q^{[k\ell]}) = \delta^{[ij][k\ell]}, \quad (\text{D.3})$$

where the symmetrized delta symbols are defined as  $\delta^{(ij)(k\ell)} = \frac{1}{2}(\delta^{ik} \delta^{j\ell} + \delta^{i\ell} \delta^{jk})$  and  $\delta^{[ij][k\ell]} = \frac{1}{2}(\delta^{ik} \delta^{j\ell} - \delta^{i\ell} \delta^{jk})$ . The traces (D.3) also show that these generators are normalized as in appendix C. Thus, comparing (3.1) and (D.2) gives the central charge

$$k_{\text{free}} = 1 \quad (\text{D.4})$$



of the  $U(n)$  flavor symmetry of  $n$  free half-hypermultiplets.

Say we have a global symmetry  $H$  with conserved currents  $J_\mu^A$  and the current-current OPE (3.1) has central charge  $k_H$ . Consider a simple subalgebra  $G \subset H$ . Similar considerations show that the central charge of the  $G$  currents,  $J_\mu^a$ , are calculated by

$$k_G = k_H I_{G \hookrightarrow H}. \tag{D.5}$$

We now compute the contribution of  $n$  half-hypermultiplets to the central charge of a weakly gauged subgroup  $G \subset U(n)$  of the  $U(n)$  (free) flavor symmetry. Suppose  $G$  is such that the  $n$  half-hypermultiplets, which form the  $\mathbf{n}$  of  $U(n)$ , transform as  $\mathbf{n} = \oplus_i \mathbf{r}_i$  under  $G$ . (D.5), (D.4), and (C.1) then imply

$$k_G = k_{\text{free}} I_{G \hookrightarrow U(n)} = \frac{1}{T(\mathbf{n})} \sum_i T(\mathbf{r}_i) = \sum_i T(\mathbf{r}_i). \tag{D.6}$$

This is the expected result: the contribution of half-hypers to the beta function is proportional to the sum of the indices of the representations of the half-hypers.

Finally, we compute the central charges of the  $U(1)_R \times SU(2)_R$  symmetry for free half-hypermultiplets and vector multiplets. A free half-hypermultiplet in a scale-invariant  $N = 2$  gauge theory has a complex scalar with  $R = 0$  and  $(I, I_3) = (1/2, 1/2)$  and a Weyl fermion with  $R = 1$  and  $I = 0$ , which, together with (D.1) and (3.1), gives  $k_{U(1)_R\text{-half-hyper}} = 2/3$ . A free vector multiplet has an  $R = 2$ ,  $I = 0$  complex scalar, a doublet of  $R = 1$ ,  $I = 1/2$  Weyl fermions, and an  $R = 0$ ,  $I = 0$  vector field, giving  $k_{U(1)_R\text{-vector}} = 8/3$ . Similarly,  $k_{SU(2)_R\text{-half-hyper}} = 1/12$ , and  $k_{SU(2)_R\text{-vector}} = 1/3$ .

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